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Pták Sum of a Boolean Algebra with an Effect Algebra and Its Completeness

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The aim of the present paper is to show that a bounded Boolean power of an effect algebra has all the analogous properties required for Pták's sum of a Boolean algebra and an orthomodular lattice and to prove a theorem about its completeness. We also give for elements of that Pták sum an important form for their expression.

1. BOOLEAN POWER OF AN EFFECT ALGEBRA AS A PTÁK SUM

In the axiomatic approach to quantum mechanics, the event structure of a physical system is a quantum logic (Pták and Pulmannová, 1981). Recently there has appeared a new axiomatic model, a difference poset (Kôpka and Chovanec, 1994) which is in some sense an effect algebra (Foulis and Bennett, 1994) representing unsharp measurements or observations on a physical system.

Definition 1.1. Let $(P; \oplus, 0, 1)$ be a system consisting of a set P with two special elements $0, 1 \in P$ and equipped with a partially defined binary operation $\oplus$ satisfying the following conditions for all $p, q, r \in P$:

(i) $p \oplus q = q \oplus p$ if one side is defined.
(ii) $p \oplus (q \oplus r) = (p \oplus r) \oplus q$ if one side is defined.
(iii) For every $p \in P$ there exists a unique $q \in P$ such that $p \oplus q = 1$.
(iv) If $1 \oplus p$ is defined, then $p = 0$.

Then $(P; \oplus, 0, 1)$ is called an *effect algebra*.

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In every effect algebra we introduce the partial ordering via $a \leq b$ iff there exists $c \in P$ with $a \oplus c = b$ and the partially defined binary operation $\ominus$ via $b \ominus a$ is defined and $b \ominus a = c$ iff $a \oplus c$ is defined and $a \oplus c = b$, for all $a, b, c \in P$.

From now on, we make the assumptions that $(B; \vee, \wedge, 0_B, 1_B)$ is a Boolean algebra and $(P, \oplus_P, 0_P, 1_P)$ is an effect algebra and we denote them briefly B and P. According to Burris (1975), we shall call a *bounded Boolean power* (of P by B) the effect algebra which has as its universe the set

$$P[B]^* = \{f \in B^P | f(P) \text{ is a finite subset of } B$$

$$\text{with } f(P) = 1_B \text{ and } f(t_1) \wedge f(t_2) = 0_B \text{ for all } t_1 \neq t_2, t_1, t_2 \in P\}$$

The partial binary operation $\oplus$ on $P[B]^*$ is defined as follows:

For $f, g \in P[B]^*, f \oplus g$ is defined iff for all $a, b \in P$ with $f(a) \wedge g(b) \neq 0_B$ the operation $a \oplus_P b$ is defined, in which case

$$f \oplus g(t) = \vee\{f(a) \wedge g(b) | a, b \in P \text{ with } a \oplus_P b = t\}, \qquad t \in P$$

Moreover,

$$0(0_P) = 1_B \text{ and } 0(t) = 0_B \text{ for all } t \neq 0_P$$

$$1(1_P) = 1_B \text{ and } 1(t) = 0_B \text{ for all } t \neq 1_P$$

We leave to the reader the verification that $(P[B]^*, \oplus, 0, 1)$ is an effect algebra. We also will leave to the reader the verification that the partial order on $P[B]^*$ is defined via

$$f \leq g \qquad \text{iff} \quad \vee\{f(a) \wedge g(b) | a, b \in P \text{ with } a \leq b\} = 1$$

and the difference operation $\ominus$ (associated to $\oplus$) is as follows:

$f \ominus g$ is defined iff $f \leq g$, in which case $(f \ominus g)(t) = \vee\{f(a) \wedge g(b) | a, b \in P \text{ with } a \ominus_P b = t\}$, $t \in P$.

If B is complete, then we can omit in the definition the requirement that $f(P)$ is finite; we obtain a *Boolean power* $P[B]$ (Burris, 1975).

It is easily seen that for every $f \in P[B]^*$ there exist a uniquely defined $n \in N$, mutually orthogonal nonzero elements $a_1, a_2, \ldots, a_n \in B$ with $\vee\{a_k | k = 1, 2, \ldots, n\} = 1$, and mutually different elements $b_1, b_2, \ldots, b_n \in P$ such that $f(b_k) = a_k$ for $k = 1, 2, \ldots, n$ and $f(t) = 0_B$ for every $t \in P \setminus \{b_1, b_2, \ldots, b_n\}$. According to that we shall use (for brevity) the notation $[(a_1, b_1), \ldots, (a_n, b_n)]$ instead of the definition of f and in that case we shall write $f = [(a_1, b_1), \ldots, (a_n, b_n)]$. Thus we have $1 = [(1_B, 1_P)]$ and $0 = [(1_B, 0_P)]$. It is routine to show that the maps

$$\varphi: a \in B \rightarrow \varphi(a) = [(a, 1_P), (a', 0_P)] \in P[B]^*$$

$$\psi: b \in P \rightarrow \psi(b) = [(1_B, b)] \in P[B]^*$$

are embeddings which preserve all suprema and infima existing in B, resp. P.

Proposition 1.1. If for elements a, b of an effect algebra P there exist $a \vee b$, $a \wedge b$, and $a \oplus_P b$, then $a \oplus_P b = (a \vee b) \oplus_P (a \wedge b)$.

See Riečanová (n.d.) for the proof.

Proposition 1.2. For all $0_B \neq a \in B$, $b \in P$ there exists $\varphi(a) \wedge \psi(b)$. Moreover, if $a, c \in B$ are such that $a \wedge c = 0_B$ and $b, d \in P$, then there exists

$$(\varphi(a) \wedge \psi(b)) \vee (\varphi(c) \wedge \psi(d)) = (\varphi(a) \wedge \psi(b)) \oplus (\varphi(c) \wedge \psi(d))$$

Proof. It is easy to see that

$$\begin{aligned}
&(\varphi(a) \wedge \psi(b)) \vee (\varphi(c) \wedge \psi(d)) \\
&\quad = [(a, b), (a', 0_P)] \vee [(c, d), (c', 0_P)] \\
&\quad = [(a' \wedge c, 0_P \vee d), (a \wedge c', b \vee 0_P), (a' \wedge c', 0_P)] \\
&\quad = [(a, b), (c, d), (a' \wedge c', 0_P)]
\end{aligned}$$

if $a' \wedge c' \neq 0_B$ and it is equal to $[(a, b), (c, d)]$ if $a' \wedge c' = 0_B$. Moreover,

$$\begin{aligned}
&1 \ominus \varphi(c) \wedge \psi(d) \\
&\quad = [(1_B, 1_P)] \ominus [(c, d), (c', 0_P)] \\
&\quad = [(c, 1_P \ominus_P d), (c', 1_P \ominus_P 0_P)] \\
&\quad \geq [(a, b), (a', 0_P)] \\
&\quad = \varphi(a) \wedge \psi(b)
\end{aligned}$$

which implies that $(\varphi(a) \wedge \psi(b)) \oplus (\varphi(c) \wedge \psi(d))$ exists. Using Proposition 1.1, we have

$$\begin{aligned}
&(\varphi(a) \wedge \psi(b)) \oplus (\varphi(c) \wedge \psi(d)) \\
&\quad = (\varphi(a) \wedge \psi(b)) \vee (\varphi(c) \wedge \psi(d))
\end{aligned}$$

since $\varphi(a) \wedge \varphi(c) = 0$.

Proposition 1.3. Let $f = [(a_1, b_1), \ldots, (a_n, b_n)] \in P[B]^*$. Then

$$\begin{aligned}
f &= \vee\{\varphi(a_k) \wedge \psi(b_k) \mid k = 1, 2, \ldots, n\} \\
&= (\varphi(a_1) \wedge \psi(b_1)) \oplus \cdots \oplus (\varphi(a_n) \wedge \psi(b_n))
\end{aligned}$$

Proof. It is routine to show that $f = [(a_1, b_1), \ldots, (a_n, b_n)] = \vee\{\varphi(a_k) \wedge \psi(b_k) \mid k = 1, \ldots, n\}$. The remainder of the statement for $n = 1$ and $n = 2$ follows from Proposition 1.2. We can proceed by induction. Suppose that the statement holds for some $n \geq 2$. Let $g = [(a_1, b_1), \ldots, (a_n, b_n), (a_{n+1}, b_{n+1})] \in P[B]^*$. Using the de Morgan law, we have

$$1 \ominus \vee\{\varphi(a_k) \wedge \psi(b_k) \mid k = 1, \ldots, n\}$$
$$= \wedge\{1 \ominus \varphi(a_k) \wedge \psi(b_k) \mid k = 1, \ldots, n\}$$
$$\geq \varphi(a_{n+1}) \wedge \psi(b_{n+1})$$

since $\varphi(a_{n+1}) \leq 1 \ominus \varphi(a_k)$, $k = 1, \ldots, n$, implies

$$\varphi(a_{n+1}) \wedge \psi(b_{n+1}) \leq \varphi(a_{n+1}) \leq 1 \ominus \varphi(a_k) \leq 1 \ominus \varphi(a_k) \wedge \psi(b_k)$$
$$k = 1, \ldots, n.$$

Thus there exists

$$(\vee\{\varphi(a_k) \wedge \psi(b_k) \mid k = 1, \ldots, n\}) \oplus (\varphi(a_{n+1}) \wedge \psi(b_{n+1}))$$
$$= \vee\{\varphi(a_k) \wedge \psi(b_k) \mid k = 1, \ldots, n + 1\}$$

since $\varphi(a_{n+1}) \wedge (\vee\{\varphi(a_k) \mid k = 1, \ldots, n\}) = 0$.

Every *homomorphism* $m: P \to \langle 0, 1\rangle$ [i.e., $m(a \oplus_P b) = m(a) + m(b)$ for all $b, a \in P$ with existing $a \oplus_P b$] has the properties $m(0_P) = 0$ and for all orthogonal pairs of elements $a, b \in P$ (i.e., $a \oplus_P b$ exists and $a \wedge b = 0_P$), if $a \vee b \in P$, then $m(a \vee b) = m(a \oplus_P b) = m(a) + m(b)$. If, moreover, $m(1_P) = 1$, then we call m a *state* on P.

Proposition 1.4. If $s: B \to \langle 0, 1\rangle$ and $m: P \to \langle 0, 1\rangle$ are states, then $\mu: P[B]^* \to \langle 0, 1\rangle$ defined for every $f = [(a_1, b_1), \ldots, (a_n, b_n)] \in P[B]^*$ by $\mu(f) = s(a_1)\cdot m(b_1) + \cdots + s(a_n)\cdot m(b_n)$ is a state on $P[B]^*$.

Proof. Suppose that $f = [(a_1, b_1), \ldots, (a_n, b_n)]$, $g = [(c_1, d_1), \ldots, (c_m, d_m)] \in P[B]^*$ with existing $f \oplus g$. Then

$$f \oplus g = [(a_i \wedge c_j, b_i \oplus_P d_j)]_{\substack{i,j \\ a_i \wedge c_j \neq 0_B}}$$

and

$$\mu(f \oplus g) = \sum_{\substack{i,j \\ a_i \wedge c_j \neq 0_B}} s(a_i \wedge c_j)\cdot m(b_i \oplus_P d_j)$$
$$= \sum_{\substack{i,j \\ a_i \wedge c_j \neq 0_B}} s(a_i \wedge c_j)(m(b_i) + m(d_j))$$

Since $s(a_i) = \sum_{j=1}^{m} s(a_i \wedge c_j)$ and $m(c_j) = \sum_{i=1}^{n} s(a_i \wedge c_j)$, we conclude that $\mu(f \oplus g) = \mu(f) + \mu(g)$. Evidently $\mu(1) = 1$.

We see that in view of the proved propositions, $P[B]^*$ has properties analogous to those required for the Pták sum of a Boolean algebra and an orthomodular lattice (Pták, 1986).

2. COMPLETENESS

It is known that also for two complete Boolean algebras B_1, B_2 the bounded Boolean power $B_1[B_2]^*$ need not be complete. In this section we prove the following statement (we follow the notation of Section 1):

Theorem 2.1. The bounded Boolean power $P[B]^*$ of an effect algebra P by a Boolean algebra B is a complete lattice if and only if both P and B are complete and at least one of them is finite.

We have divided the proof into a sequence of lemmas and propositions.

Lemma 2.2. Suppose that $K \subset B$ with $\vee K$ existing in B and $d \in P$. Then

$$\varphi(\vee K) \wedge \psi(d) = \vee\{\varphi(a) \wedge \psi(d) \mid a \in K\}$$

Proof. Let $c = \vee K$. By the definitions of φ and ψ we have $\varphi(\vee K) = [(\vee K, 1_P), ((\vee K)', 0_P)]$, $\psi(d) = [(1_B, d)]$. Suppose that $y = [(a_1, b_1), \ldots, (a_n, b_n)] \geq \varphi(a) \wedge \psi(d) = [(a, d), (a', 0)]$, for every $a \in K$. If, for $k \in \{1, \ldots, n\}$, $(\vee K) \wedge a_k \neq 0_B$, then there exists $a \in K$ with $a \wedge a_k \neq 0_B$ and then $d \leq b_k$. Thus $\varphi(\vee K) \wedge \psi(d) \leq y$. We conclude that $\varphi(\vee K) \wedge \psi(d) = \vee\{\varphi(a) \wedge \psi(d) \mid a \in K\}$.

Lemma 2.3. If an effect algebra P_1 is a supremum-dense subalgebra of an effect algebra P_2, then all suprema and infima existing in P_1 are inherited for P_2.

We refer the reader to Riečanová (n.d.), Theorem 1.7, for the proof.

For every Boolean algebra B its MacNeille completion (i.e., completion by cuts) is, up to a unique isomorphism over B, a complete Boolean algebra $\overline{B}$ into which B can be supremum-dense embedded (i.e., every element of $\overline{B}$ is a supremum of some elements of B) (Schmidt, 1956). Moreover, the embedding α preserves all suprema and infima existing in B. We usually identify $\alpha(B) \subseteq \overline{B}$ with B. In this sense $P[B]^*$ is a subalgebra of $P[\overline{B}]^*$ (Burris, 1975, Proposition 2.3).

Proposition 2.4. $P[B]^*$ is supremum-dense in $P[\overline{B}]^*$.

Proof. Let $f = [(a_1, b_1), \ldots, (a_n, b_n)] \in P[\overline{B}]^*$. Since B is supremum-dense in $\overline{B}$, there exist $M_k \subseteq B$ with $\vee M_k = a_k$, $k = 1, \ldots, n$. Thus by Lemma 2.2, $\varphi(a_k) \wedge \psi(b_k) = \varphi(\vee M_k) \wedge \psi(b_k) = \vee\{\varphi(c) \wedge \psi(b_k) \mid c \in M_k\}$, $k = 1, \ldots, n$. It follows that $f = \vee\{\varphi(c) \wedge \psi(b_k) \mid c \in M_k, k = 1, \ldots, n\}$.

Similar arguments apply to the case $P[\overline{B}]$; we can prove the following assertion:

Proposition 2.5. $P[B]^*$ is supremum-dense in $P[\overline{B}]$.

Proposition 2.6. If $P[B]^*$ is complete, then P and B are complete.

Proof. (1) Let $M \subseteq P$. Let us put $D = \{d \in P \mid d \geq b$ for every $b \in M\}$. For every $d \in D$ let $g_d = [(1_B, d)]$. Completeness of $P[B]^*$ implies that there exists $f = \wedge\{g_d \mid d \in D\}$. Since $\vee\{f(t) \mid t \in P\} = 1_B$, there exists $t_0 \in P$ with $f(t_0) \neq 0_B$. It follows that $f(t_0) \wedge 1_B \neq 0_B$ and hence $t_0 \leq d$ for every $d \in D$. Moreover, $f \geq g_d \geq \psi(b) = [(1_B, b)]$ implies $t_0 \geq b$ for every $b \in M$. We conclude that $t_0 = \vee M \in P$.

(2) By Proposition 2.4, completeness of $P[B]^*$ implies that $P[B]^* = P[\overline{B}]^*$, using also Lemma 2.3. Thus for any $K \subseteq B$ there exists $\vee K \in \overline{B}$ and $[\vee K, 1_P), (\vee K)', 0_P)] \in P[\overline{B}]^* = P[B]$, which implies that $\vee K \in B$.

Using the de Morgan laws we conclude that P and B are complete lattices.

Proposition 2.7. If $P[B]^*$ is complete, then $P[B]^* = P[\overline{B}]^* = P[\overline{B}]$ and at least one of P and B is finite.

Proof. The completeness of $P[B]^*$ implies $B = \overline{B}$ by Proposition 2.6. In view of Proposition 2.5 and Lemma 2.3 we obtain $P[B]^* = P[\overline{B}]^* = P[\overline{B}]$. Hence at least one of P and B is finite.

Proposition 2.8. If P and B are complete and at least one of them is finite, then $P[B]^*$ is complete.

Proof. (1) Suppose that B is complete and $P = \{d_1, \ldots, d_n\}$. Let $M \subseteq P[B]^*$. For $i = 1, \ldots, n$ let us put $K_i = \{a \in B \mid \varphi(a) \wedge \psi(d_i) \leq f, f \in M\}$ and $M_i = \{\varphi(a) \wedge \psi(d_i) \mid a \in K_i\}$. Since B is complete, there exists $\wedge K_i \in B$ and by Lemma 2.2 we have $\varphi(\vee K_i) \wedge \psi(d_i) = \vee\{\varphi(a) \wedge \psi(d_i) \mid a \in K_i\} = \vee M_i \in P[B]^*$. Since P is a lattice, $P[B]^*$ is a lattice, too, and thus $\vee M = \vee\{\vee M_i \mid i = 1, \ldots, n\} \in P[B]^*$.

(2) Suppose that B is finite and A is the set of all atoms of B. Then $P[B]^*$ is isomorphic to the direct product $\Pi\{P_a \mid a \in A\}$, where $P_a = P$ for every $a \in A$. It follows that $P[B]^*$ is complete if P is complete.

Now the proof of Theorem 2.1 follows by Propositions 2.6–2.8.

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